Chapter 8

Alpha Decay

8.1 Definitions and Energetics

Alpha decay occurs spontaneously from the ground state of many heavy nuclei with $A > 208$ and in some neutron deficient rare earth nuclei, see Fig. 8.1.[1] The $\alpha-$decay **rates** vary from μs to the age of the universe.

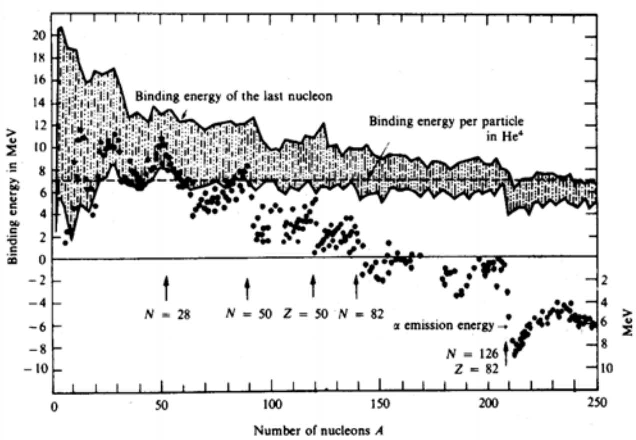

Figure 8.1: The dots are the $\alpha-$binding energy in a numbers of nuclides. Above $A \sim 150$ the $\alpha-$separation energy is negative and spontaneous $\alpha-$decay occurs.

Example:

$$^{238}_{92}\mathrm{U}_{146} \rightarrow \ ^{134}_{90}\mathrm{Th}_{144} + \ ^{4}_{2}\mathrm{He}_{2} + Q_\alpha.$$

The total energy release

$$\begin{aligned} Q_\alpha &= E_{(^4\mathrm{He})} + E_{(^{134}\mathrm{Th})} + \mathrm{Excitation} \\ &= \Delta M(^{238}\mathrm{U}) - \Delta M\left(^{234}\mathrm{Th}\right) - \Delta M\left(^{4}\mathrm{He}\right) \\ &= 47.307 - 40.612 - 2.425 = 4.270 \text{ MeV}. \end{aligned}$$

From energy and momentum conservation we find the relation between Q_α and the kinetic energy of the $\alpha-$particle, E_α. Let us assume (as is a very good assumption for e-e parents, and daughters) that there is little internal excitation of the daugher.[2] Before decay the momentum is 0 so the α-particle and daughter momenta must cancel.

$$\begin{aligned} \vec{\mathbf{p}}_\alpha &= -\vec{\mathbf{p}}_D \\ |\vec{\mathbf{p}}_\alpha| &= |\vec{\mathbf{p}}_D| \end{aligned}$$

$$\begin{aligned} m_\alpha v_\alpha &= M_D v_D, v_\alpha = \frac{M_D}{m_\alpha} v_D, \\ \frac{1}{2}\left(m_\alpha v_\alpha\right)^2 &= \frac{1}{2}\left(M_D v_D\right)^2, \\ m_\alpha E_\alpha &= M_D E_D \end{aligned}$$

From energy conservation

$$E_\alpha + E_D = Q_\alpha,$$

thus

$$\begin{aligned} m_\alpha E_\alpha &= M_D E_D = M_D\left(Q_\alpha - E_\alpha\right), \text{ or} \\ \left(m_\alpha + M_D\right) E_\alpha &= M_D Q_\alpha, \text{ and} \\ E_\alpha &= Q_\alpha \frac{M_D}{m_\alpha + M_D} = Q_\alpha \frac{M_D}{M_P} \\ E_D &= Q_\alpha \frac{M_\alpha}{M_P}. \end{aligned}$$

For the ^{238}U decay

$$\begin{aligned} E_\alpha &= 4.270 \frac{234}{234+4} = 4.198 \text{ MeV}, \\ E_{Th} &= 4.270 \frac{4}{234+4} = 0.072 \text{ MeV}. \end{aligned}$$

Note that this recoil energy is many times larger than the energy of chemical bonds (1-5 eV), so if U is bound to a molecule, the Th isn't.

[1] There is a connection to fission and the so called "fissility paramenter" which should be appreciated after Fission is studied.

[2] If there is, just reduce the Q-value by the exciation of the daughter, i.e. increase the mass by the excitation of the daughter.

8.2 $\alpha-$decay rates

General:

- Even-Even parents decay almost exclusively to the ground state of very low-lying excited states.
- The $\alpha-$half life is a strong function of the energy, E_α. This is shown in Fig. 8.2. Note that the half life changes by several orders or magnitude just for 1 MeV change in E_α. The lines are for fixed Z, connecting various isotopes.

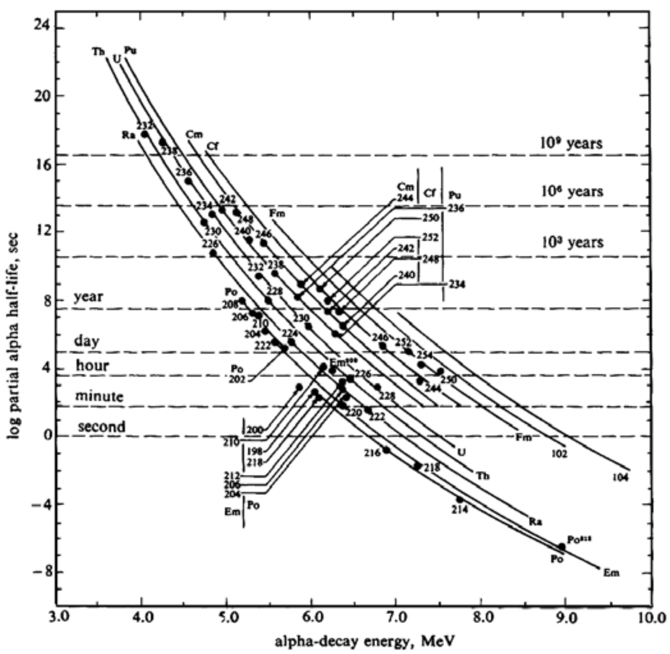

Figure 8.2: Partial $\alpha-$half lives as a function of $\alpha-$energy for even-even nuclei. The lines connect isotopes of a given Z, larger Z's going up.

In 1912 Geiger and Nutall obtained an empirical relationship relating the $t_{1/2.\alpha}$ with Q_α,

$$\log t_{1/2}(s) = a + \frac{b}{Q_\alpha^{1/2}}, \text{ with} \qquad (8.1)$$
$$a \simeq -1.61\, Z_D^{2/3} - 21.4,$$
$$b \simeq 1.61\, Z_D.$$

We shall derive this expression shortly from simple quantum-mechanical arguments.

Simple Rate theory

The $\alpha-$decay is mainly governed by the penetration of the Coulomb barrier by the α-particle. The numverical value of the maximum of the potential is called th "Coulomb barrier", B in Fig. 8.3. As $B \gg E_\alpha \approx Q_\alpha$, α-decay is classical forbidden. When the value of the Coulomb potential is equal to Q_α, then an outward classical turning point radius b is determined.

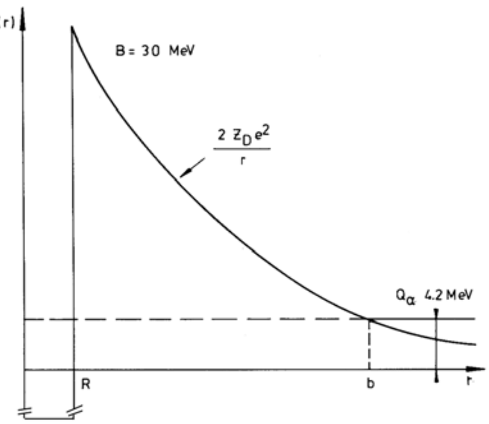

Figure 8.3: Coulomb barrier for $\alpha-$decay.

The decay rate λ_α is

$$\lambda_\alpha = \lambda_0 \cdot P,$$

where $\lambda_0 =$ is a reduced rate constant and P is the probability of penetration through the barrier. Logically λ_0 is the frequency f of attack of the barrier by an α particle preformed in the nucleus. This simple logical identification hides the complexity of λ_0. For example λ_0 contains all the structure dependent physics which we discuss later.

The frequency f can be estimated as

$$f = \frac{v_{int}}{R} = \frac{1.39\sqrt{\left(\frac{A-4}{A}E_\alpha\right)/4}\ \frac{\text{cm}}{ns} \cdot 10^{13}\frac{\text{fm}}{cm} \cdot 10^9 \frac{\text{ns}}{s}}{1.4 \cdot A^{1/3}\text{ fm}}$$
$$= 5 \cdot 10^{21} \frac{\sqrt{E_\alpha \frac{A-4}{A}}}{A^{1/3}}\ \text{s}^{-1}. \qquad (8.2)$$

For $E_\alpha = 4$ MeV and $A = 220$, $f = 1.63 \times 10^{21}$ s^{-1}, i.e. a lot of attacks per second! Since all $\alpha-$half lifes are long compared to this is, P must be extremely small ($10^{-13} - 10^{-39}$, after all classically $P = 0$). In quantum mechanics we can use the WKB approximation method to compute the barrier penetration factor.[3] This method assumes wave function forms both outside and inside the classically forbidden region, and forces the values

[3]Named after Wentzel, Kramers and Brillouin, three giants of classical and early quantum physics. Kramers is best known for this work in statistical physics and Brillouin for his work in solid state physics.

and first dervatives of the wave functions to match at the boundaries. The WKB result for the penetrability is the exponential of the integrated "momentum" $[2\mu(V_C(r)-Q_\alpha)]^{1/2}$ under the barrier.[4]

$$
\begin{aligned}
P &= \exp\left\{-\frac{2}{\hbar c}\int_R^b \left[2\mu c^2(V_C(r)-Q_\alpha)\right]^{1/2} dr\right\} \\
&= \exp\{-2G\}.
\end{aligned}
$$

G is known as the Gamow factor.

Numerical calculation for realistic barriers is readily done (and as you will do in a problem). The simple Coulomb form $V_C(r)=\frac{e^2zZ_D}{r}$ for the potential gives an accurate estimate of the outer classical turning point.

$$
\begin{aligned}
Q_\alpha &= \frac{e^2zZ_D}{b} \text{ with} \\
b &= \frac{e^2zZ_D}{Q_\alpha}.
\end{aligned}
$$

One can get a reasonable estimate for G (and thus the decay rate) by using the simple $1/r$ potential form and integrating from the nuclear radius out to b. (The potential shape is grossly inaccurate near the peak, but most of the contribution to G comes from large r, where the simple 1/r form is accurate.

$$
\begin{aligned}
G &= \frac{1}{\hbar c}\int_R^b \left\{2\mu c^2\left[\frac{e^2zZ_D}{r}-Q_\alpha\right]\right\}^{1/2} dr \\
&= \frac{1}{\hbar c}\sqrt{2\mu c^2 Q_\alpha}\int_R^b \left(\frac{b}{r}-1\right)^{1/2} dr \\
&= \frac{b}{\hbar c}\sqrt{2\mu c^2 Q_\alpha}\left\{\arccos\sqrt{\frac{R}{b}}-\sqrt{\frac{R}{b}-\left(\frac{R}{b}\right)^2}\right\} \\
\lambda_\alpha &= \frac{v_{int}}{R}\exp\{-2G\}. \qquad (8.3)
\end{aligned}
$$

The Geiger-Nutall expression can be derived from this result.[5]

Let us use the WKB result to calculate the half life for the decay

$$ {}^{238}\text{U} \rightarrow {}^{234}\text{Th} + {}^{4}\text{He} + Q_\alpha. $$

[4]The Q-value is used here as it is the cost in the CM system of particle removal. This puts this energy in the same system as V_C which is also calculated in the CM.

[5]Substitute $Q_\alpha=\frac{1}{2}\mu v^2$ in $\sqrt{2\mu Q_\alpha}=\sqrt{2\mu\frac{1}{2}\mu v^2}=\mu v$ and $b=\frac{e^2zZ_D}{Q_\alpha}=\frac{2e^2zZ_D}{\mu v^2}$ to get

$$
\begin{aligned}
G &= \frac{2e^2zZ_D}{\hbar\,\mu v^2}\mu v\left\{\arccos\sqrt{\frac{R}{b}}-\sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}}\right\} \\
&= \frac{2e^2zZ_D}{\hbar v}\left\{\arccos\sqrt{\frac{R}{b}}-\sqrt{\frac{R}{b}}\sqrt{1-\frac{R}{b}}\right\}.
\end{aligned}
$$

If $R/b \ll 1$, then $\arccos\sqrt{\frac{R}{b}}\simeq\frac{\pi}{2}-\sqrt{\frac{R}{b}}$, and $\sqrt{1-\frac{R}{b}}\simeq 1-\frac{1}{2}\frac{R}{b}\simeq 1$, and

$$ G=\frac{2e^2zZ_D}{\hbar v}\left(\frac{\pi}{2}-2\sqrt{\frac{R}{b}}\right). $$

Since $b=\frac{2e^2zZ_D}{\mu v^2}$ we get

$$
\begin{aligned}
G &= \frac{2e^2zZ_D}{\hbar v}\frac{\pi}{2}-\sqrt{\frac{16R}{}\left(\frac{2e^2zZ_D}{\hbar v}\right)^2\frac{\mu v^2}{2e^2zZ_D}} \\
&= \frac{\pi e^2zZ_D}{\hbar v}-\frac{4}{\hbar}\left(2\mu e^2zZ_DR\right)^{1/2}, \text{ for } \alpha-\text{decay}, \\
&= \frac{2\pi e^2Z_D}{\hbar v}-\frac{8}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}, \\
\lambda_\alpha &= \frac{v_{int}}{R}\exp\left[-2G\right] \\
&= \frac{v_{int}}{2R}\exp\left[-\frac{4\pi e^2Z_D}{\hbar v}+\frac{16}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}\right], \\
t_{1/2} &= \ln 2\frac{R}{v_{int}}\exp\left[\frac{4\pi e^2Z_D}{\hbar v}-\frac{16}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}\right], \text{ and} \\
\ln t_{1/2} &= \ln\left(\ln 2\frac{R}{v_{int}}\right)\left[\frac{4\pi e^2Z_D}{\hbar v}-\frac{16}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}\right] \\
&= \ln\left(\ln 2\frac{R}{v_{int}}\right)\left[\frac{4\pi e^2Z_D}{\hbar\sqrt{2Q_\alpha/\mu}}-\frac{16}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}\right] \\
&= a+\frac{b}{\sqrt{Q_\alpha}}, \text{ with} \\
a &= -\ln\left(\ln 2\frac{R}{v_{int}}\right)\frac{16}{\hbar}\left(\mu e^2Z_DR\right)^{1/2}, \text{ and} \\
b &= \ln\left(\ln 2\frac{R}{v_{int}}\right)\frac{4\pi e^2Z_D\sqrt{\mu}}{\hbar\sqrt{2}}.
\end{aligned}
$$

This is the Geiger-Nutall expression.

Here $E_\alpha=4.198$ MeV, $Q_\alpha=\frac{238}{234}\cdot 4.198=4.270$ MeV and $\mu_\alpha=\frac{4\times 234}{234+4}=3.932$. A reasonable value of the inner turning point is $R=1.2\cdot 234^{1/3}+1.6=7.4+1.6$ fm $=9.0$ fm.

$$
\begin{aligned}
v_{int} &= \sqrt{\frac{2\times 4.270}{\mu_\alpha}} \\
&= 3\cdot 10^{23}\frac{\text{fm}}{s}\times\sqrt{\frac{2\times 4.270}{3.932\times 931.5}}=1.449\cdot 10^{22}\frac{\text{fm}}{s}, \\
f &= \frac{v_{int}}{R}=\frac{1.449\times 10^{22}\ \frac{\text{fm}}{s}}{8.995\text{ fm}}=1.611\times 10^{21}\ s^{-1}.
\end{aligned}
$$

$$
\begin{aligned}
b &= \frac{1.44\times 2\times 90}{4.270}=60.7025\text{ fm}, \\
\frac{R}{b} &= \frac{8.995\text{ fm}}{60.7025\text{ fm}}=0.14818,\ \sqrt{\frac{R}{b}}=0.38494, \\
G &= \frac{60.70\text{ fm}}{197.3\text{ MeVfm}}\sqrt{2\cdot 3.93(931.5)\cdot 4.270} \\
&\quad \times\left[\arccos(0.385)-0.385\cdot\sqrt{1-0.148}\right] \\
&= 54.41\cdot(1.176-0.355)=44.64 \\
\lambda_\alpha &= 1.611\cdot 10^{21}\ s^{-1}\exp\left(-2\cdot 44.6\right) \\
&= 2.734\times 10^{-18}\ \text{s}^{-1}
\end{aligned}
$$

$$t_{1/2,\alpha} = \frac{\ln 2}{\lambda_\alpha} = \frac{\ln 2}{2.734 \cdot 10^{-18}} = 2.54 \cdot 10^{17}\text{s}$$
$$= 2.54 \cdot 10^{17} \frac{1}{3600 \times 24 \times 365.25} = 8.03 \cdot 10^9 \text{ y.}$$

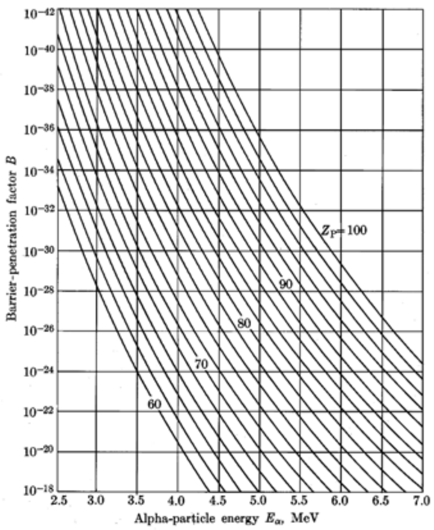

Figure 8.4: Barrier penetration factors calculated numerically.

The experimental value is $4.5{\cdot}10^9$ y, not bad at all given the approximations involved. The overall effect of increasing Z and increasing Q-value, and thus E_α is reproduced by the WKB logic, as is shown in its exercise, Fig 8.4 which should be compared to Fig. 8.2

8.2.1 Centrifugal barrier effects

In some $\alpha-$decays, the $\alpha-$particle takes away orbital angular momentum. The Coulomb "potential" is then supplemented by a misnamed centrifugal barrier,see Fig. 8.5.[6]

$$V(r) = \frac{z(Z-z)e^2}{r} + \frac{\hbar^2 \ell(\ell+1)}{2\mu_\alpha r^2},$$

Let the ratio of the centrifugal to the Coulomb barrier at $r = R$ be σ, then

$$\sigma = \frac{\hbar^2 \ell(\ell+1)}{2\mu_\alpha R^2} \cdot \frac{R}{z(Z-z)e^2}$$
$$= \frac{\hbar^2 \ell(\ell+1) R}{e^2\ z\ Z_D 2\mu_\alpha R^2}, \text{ let } z = 2$$
$$\sigma = \frac{\hbar^2 \ell(\ell+1)}{4e^2\ Z_D \mu_\alpha R}.$$

[6] It is not a true barrier or potential, it is rather just an energy which is not in the radial direction and thus not available for the penetration coordinate. Either way this energy must be "spent" on particle removal at finite l.

For ^{238}U we have $R = 1.2 \cdot 234^{1/3} + 1.6$ fm $= 8.995$ fm.

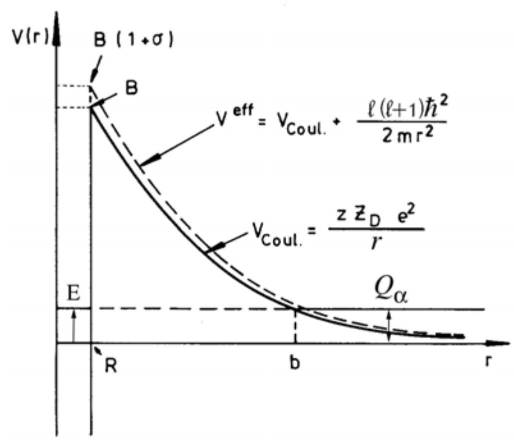

Figure 8.5: Coulomb and centrifugal barriers as a function of radius r.

$Z_D = 90, Q_\alpha = 4.270$ MeV, and $\mu_\alpha = 3.932$ amu. Then

$$\sigma = \frac{6.465^2 \text{ MeV} \cdot \text{amu} \cdot \text{fm}^2 \ell(\ell+1)}{4 \cdot 1.44 \text{ MeV} \cdot \text{fm} \cdot 90 \cdot 3.932\text{amu} \cdot 8.995\text{fm}}$$
$$= 0.00228 \cdot \ell(\ell+1).$$

The expression $\frac{Q_\alpha}{B} = \frac{R}{b}$ becomes

$$\frac{Q_\alpha}{B(1+\sigma)} = \frac{R}{b}, \text{ and}$$

$$\sqrt{\frac{R}{b}} = \sqrt{\frac{Q_\alpha}{B(1+\sigma)}} \simeq \sqrt{\frac{Q_\alpha}{B}}\left(1 - \frac{\sigma}{2}\right).$$

Table 8.1: Ratio of λ_α probability for various ℓ values

ℓ	**0**	**1**	**2**	**3**	**4**	**5**	**6**
$\frac{\lambda_\alpha(\ell)}{\lambda_\alpha(\ell=0)}$	1	0.7	0.37	0.14	0.037	$7{\cdot}10^{-3}$	$1{\cdot}10^{-3}$

While angular momentum effect is small compared to the primary energy dependence, the reduction, sometimes called retardation, of the decay rate for finite removed angular momentum is significant, see Table 8.1 and the example shown in Fig. 8.6. One must appreciate that becasue the intrinsic spin of the α particle is zero, concervation of angular momentum requried that any difference in the spin between the parent and daughter must be matched by the orbital angular momentum ℓ of the removed α particle. That is,

$$|I_p - I_d| \leq l \leq I_p + I_d,$$

where I_p and I_d are spins of initial (parent) and final (daughter) nuclear states. For the decay of Fig. 8.6 $I_p = 0$, so $I_d \leq \ell \leq I_d$, or simply $\ell = I_d$. There is also no parity change for even orbital angular momentum $\ell = 0, 2, 4, 6$, and 8 $[\Delta\pi = (-1)^\ell]$. However. the lowest possible ℓ-wave is always favored as more anergy is reseved for the radial motion, (through the barrier rather than tangential to it.) The problem can be turned around. That is, the the comparison of the measured decay rates and those calculated with reasonable barrier shapes (as a function of l) can often provide a good guess of either the initial or final spin, if other is known.

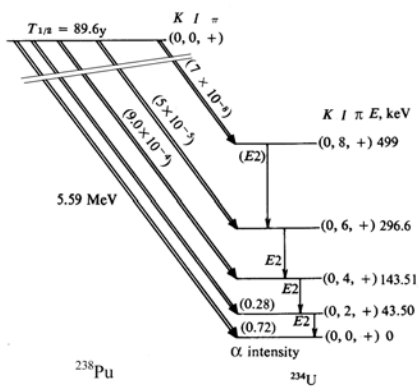

Figure 8.6: Decay scheme for ^{238}Pu. The decrease in the α-feeding of the higher lying states in the rotational band is primarily due to the decreasing energy available in the "decay channel", i.e. for penetration.

8.2.2 Nuclear structure effects

If one is not dealing with an e-e parent (and thus daughter the daughter is also not e-e) the nuclear structure effects can be pronounced. Take for example the α-decay of ^{249}Cf, see Fig. 8.7 The parent ground state spin is $9/2^-$ while that of the daughter is $7/2^+$. The ground-state to ground-state transition must be $\ell = 1\hbar$. However, decay to the excited state at 388 keV can proceed via $\ell = 0\hbar$. These two decays both occur (as do decays to several other levels) but the "lion's" share of the decay goes to the 388 keV excited state.

Barrier penetration factors are readily calculated by numerically integrating Eq. 8.3 for any barrier shape (Z and L) for any value of E_α. It is often useful to define a **reduced decay constant** λ_0 and **reduced width** $\Delta E_{0,\alpha}$ by removing the expected (smooth) dependence (on Z and $\ell-$wave). The latter, defined using the uncertainty relation is $\Delta E_{0,\alpha} = \frac{\hbar}{\tau_0} = \hbar\lambda_0 = \hbar\frac{\lambda_\alpha}{P}$. These reduced widths (see Fig. 8.8) depend on specific nuclear

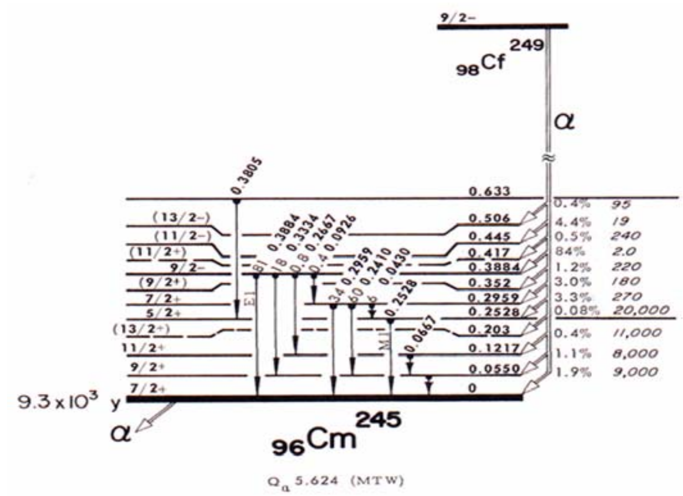

Figure 8.7: ^{249}Cf α-decay.

structure information. For example, as the calculated potentials are spherical, the reduced widths continain information on deformation of the emittor. (Deformation reduces the average barrier thus increasing the decay rate.)

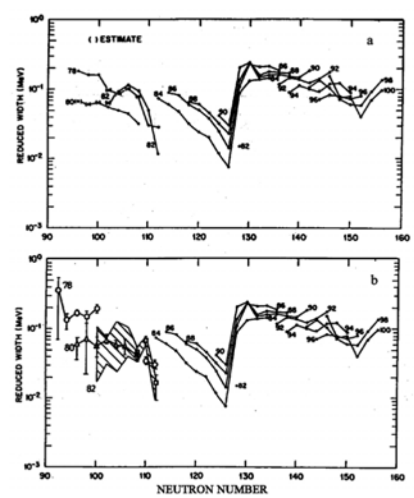

Figure 8.8: Some $\alpha-$reduced widths for even-even nuclei in the vicinity of ^{208}Pb.